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# Open Set in Metric Space

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**ABSTRACT**
In many branches of mathematics specially geometry and analysis-it has been search out that the a notion of distance function is frequently applicable in abstract sets.The ward metric is used for measurement that is based on metres , centimeter, gram and litres etc. The notation of distance function is known as metric space is nothing more than a non-empty set equipped with a concept of distance which is suitable for the treatment of convergent sequences in the set and continuous functions defined on the set. Our purpose in this paper is to develop in a systematic manner the main elementary facts about open sets in metric space. These facts are important for their own sake, and also for the sake of the motivation they provide for our later work on topological spaces.

A metric space is a set in which we can talk of the distance between any two of its elements. The definition of metric space imposes certain natural conditions on the distance between the points. Most of the analysis (real or complex) depends upon two fundamental concepts, namely, convergence of the sequences and continuity of the functions. These two concepts depends heavily on the properties of the distance between two points. We want to generalize the concept of distance for any non-empty set so that the concepts of convergent sequences and continuous functions can be generalized. This generalization gave birth to the concept of metric spaces

**KEYWARDS-** Metric space, distance ,open set , interior point

Metric Space: Let X be a non-empty set. A function d: X $\times$ X -> R, where R is the set of real numbers, is said to be a metric (or distance function) on X if it satisfies the following conditions:

i. $d(x, y) \geq 0 \ \forall x, y \in X$ (Non-negativity)
ii. $d(x, y) = 0 \iff x = y$ where $x, y \in X$
iii. $d(x, y) = d(y, x) \ \forall x, y \in X$ (Symmetry)
iv. $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$ (Triangle Inequality)

1. The ordered pair (X, d) is called a metric space and d(x, y) is called the distance between x and y. The elements of X are called its points.
2. The triangle inequality may be expressed as that the length of one side of any triangle can not exceed the sum of lengths of the other two sides.
3. The sign of equality hold in the triangle inequality when the three points are in a straight line.
4. The triangle inequality can be generalized in the following way:

$$d(x, y) \leq d(x, z_1) + d(z_1,z_2) + ..... + d(z_n, y) \text{ where } x, y, z_1, z_2......,z_n \in X.$$

**Example 1 : (Usual metric space) :** Let X = R be the set of real numbers. For $x, y \in X$. Then (X, d) is a metric space where $d(x,y) = |x-y|$ for all $x, y \in X$

**Example 2 : (Discrete metric space) :** Let X be any non-empty set. For $x, y \in X$ ,

Define $d(x, y) = \begin{Bmatrix} 0 \ if \ x = y \\ 1 \ if \ x \neq y \end{Bmatrix}$

Then (X, d) is a metric space. The metric d is called the discrete metric and the space (x, d) is called the discrete metric space.
Here these distance function satisfy all the condition of metric space.

**Distance Between Point and Subset:** Let (X,d) be a metric space and A be a subset of X. If $x \in X$ then the distance of x from A is defined as:

$$d (x, A) = \inf \{d (x,a) : a \in A\}$$

i.e., it the g.l.b. of the distance from x to the point of A.

For example, if d is usual metric for R such that d (x, y) = $|x - y|$ and if A = [1 , 2] then distance of the points x = $\frac{3}{2}, \frac{5}{2}$ are given by

$$d\left(\frac{3}{2}, A\right) = 0 \qquad \text{as } \frac{3}{2} \text{ belongs to A}$$

and
$$d\left(\frac{5}{2}, A\right) = \left|\frac{3}{2} - 2\right| = |.5| = .5 = \frac{1}{2}$$

**Diameter of Subset:** Let (X,d) be a metric space and let A be any non empty subset of X. Then the diameter of A, denoted by $\delta$(A), is defined by $\delta$(A) = sup {d(x,y): x,y $\in A\}$, i.e. the diameter of A is the supremum of the distance between the pairs of its points.

Set A is said to have finite for infinite diameter according as $\delta(A) < \infty$

for example, if d is usual metric for R such that d (x,y) = $|x-y|$ and if A=[1,3] and B= (3,6), then $\delta(A)$ = 2 and $\delta(B) = 3.$

**Distance Between Two Subsets :** Let A and B two subsets for a metric space (X,d), then distance between A and B is defined as

$$d(A,B) = \inf \{d(x,y) : x \in A, y \in B\}$$

for example, if d is usual metric for R such that d(x,y) = $|x-y|$ and if A = [1, 2], B = (2,4), then d (A,B) = 0 again of A = (1, 4] and B = inf {d(x,y) : x $\in$ A, y $\in$ B} = 2.

**Open sphere :** Let (X, d) be a metric space and Let a$\in$ X and r is any positive real number, then the set {x $\in$ X : d (x,a) < r} is called an open sphere (or open ball). The point 'a' is called the centre of the sphere and r the radius of the sphere. It is denoted by S)a,r) or $S_r$ (a) or B (a,r) or $B_r$ (a).

B (a,r) = {x $\in$ X :d (x,a) < r} = S (a,r).

Similarly a closed sphere (or closed ball) is denoted and defined by

$$S[a,r] = \{x \in X : d(x,a) \leq r\} = B[a,r].$$

Note that a ball is always non-empty since it contains at least its centre.

$x_0 \in$ X and r > 0 be any real number. The open sphere with centre at $x_0$ and radius r, is denoted by $S_r$ (x) and is defined as

$$S_r(x) = \{x \in X : d(x_0, x) < r\}$$

**1.** Open spheres and closed spheres are also known as open balls and closed balls respectively.
**2**. Other common notations used for the open sphere $S_r(x_0)$ are $S(x_0,r)$, $B_r(x_o,)$ and $B(x_0,r)$. Similarly the closed sphere $S_r[x_0]$ is also denoted by $S[x_0,r]$, $B_r[x_0]$ and $B[x_0, r)$.
**3.** A sphere (open or closed) is always non-empty since it contains its centre at least.
4. All the element in the open sphere are from the set X. So, it is always a subset of X.

**Example 1 :** In the usual metric space (R, d) show that
(i) Every open sphere is an open interval.
(ii) Every closed sphere is a closed interval.

**Solution :** We know that usual metric on R is defined as

$$d(x, y) = |x - y| \text{ for } x, y \in R$$

Let $x_0, \in$ R and r > o be any real number then

(i)
$$\begin{aligned} S_r(x_0) &= \{ x \in R: d(x_0,x) < r\} \\ &= \{ x \in R : |x-x_0|< r\} \\ &= \{ x \in R : -r < x - x_0 < r\} \\ &= \{ x \in R : x_0 -r < x < x_0 + r\} \\ &= (x_o - r, x_o +r) \end{aligned}$$

(ii) Similarly. $S_r[x_0] = [x_0 - r, x_o + r]$

**Example 2.** Describe open and closed spheres in a discrete metric space.

**Solution :** Let (X, d) be a discrete metric space where

$$d(x, y) = \begin{cases} 0 \text{ } if \text{ } x = y \\ 1 \text{ } if \text{ } x \neq y \end{cases}$$

Open sphere in discrete metric space : Let $x_o \in$ X and r > 0 be any real number.

If $r \leq 1$, then $S_r(x_0) = \{x \in X : d(x_0, x) < r\}$
$= \{x \in X : d(x_0, x) < r \leq 1\}$
$= \{x \in X : d(x_0, x) < 1\}$
$= \{x \in X: d(x_0, x) = 0\}$
$= \{x \in X : x = x_o\} = \{x_o\}$

If $r > 1$, then $S_r(x_o) = \{x \in X : d(x_o, x) < r\}$. Here $r > 1$ and $d(x_o, x) = 0$ or 1, so $d(x_o, x) < r$ for all $x \in X$. Thus, $S_r(x_o) = X$

In words, we can say that, in a discrete metric space every open sphere of radius less than or equal to 1 is a singleton and every open sphere of radius greater than 1 is whole the space X.

**Interior Point and Interior of a Set :** a Let A be any subset of a metric space (X, d), then a point 'a'$\in$ A is called interior point of A if 'a' is the centre of an open sphere and that open sphere contained in A .

For example, let X = R and A = $(-3,3) \subseteq R$, then the point a = 1 is an interior point of A w.r.t. metric $d(x,y) = |x - y|$ for all x, y $\in$ R, since there is an open sphere $S\left(1, \frac{3}{2}\right)$ centred at 1 with radius $\frac{3}{2}$ such that

$$S\left(1, \frac{5}{2}\right) = \left\{x \in R : x \left(-\frac{1}{2}, \frac{5}{2}\right)\right\}$$

Hence any point 'a' is an interior point of subset A of a metric space (X, d) if there exists an open sphere $S_r$ (a) such that

$a \in S_r(a) \subseteq A$.

The set of all interior points of A is called the interior of A and is denoted by $A^o$.

Thus $A^\circ = \{x : x$ is an interior point of $A\}$.

Alternatively, we can define $A^\circ = \cup \{S_r(x): S_r(x) \subset A\}$

If A and B are subsets of a metric space (X, d), then

(i) $(A \cap B)^\circ = A^\circ \cap B^o$

(ii) $A^\circ \cup B^\circ \subset (A \cup B)^o$

Also show that $A^o \cup B^\circ \neq (A \cup B)^o$, by means of an example.

Let A = [0, 1], and B =[1, 2], then

$A^\circ = [0, 1]^\circ = (0, 1)$ and $B^o = (1, 2)$

$A^o \cup B^\circ = (0, 1) \cup (1, 2)$

$= \{x \in R: 0 < x < 1 \text{ or } 1 < x < 2\} = (0, 2) - (1)$

Also $A \cup B = [0,2]$

$(A \cup B)^\circ = [0, 2]^\circ = (0,2)$

Evidently $(A \cup B)^o \neq A^o \cup B^o$.

**Open set :** Let (x, d) be a metric space. A subset G of X is an open set if it is neighborhood of each of its points.

OR

A subset G of X is said to be an open set if there exists a real number r > 0 such that $S_r(x) \subseteq G$, for every $x \in G$

(i) In discrete metric space (X, d) every subset of X is open. In particular, every singleton {x} is open.

(ii) Every open interval (a, b) is an open set.

(iii) Any of intervals (a, b), (a, b), (a, b) is not an open set.

(iv) R is an open set.

(v) The sets N, Z, $\varrho$ and I are not open sets.

(vi) The set $\{1, \frac{1}{2}, \frac{1}{3}, \dots \dots\}$ is not an open set.

(vii) Any finite subset and in particular any singleton of R is not an open set.

are open sets,

**Theorem :** Let (X, d) be a metric space. Then the empty set $\Phi$ and the whole space X

Proof: To so that $\Phi$ is open $A = \{\Phi\}$

If $x \in A$ then $S_r(x)$ is any open sphere with centre x and radius r , is not exit without centre so

$S_r(x) = \Phi \subseteq \{\Phi\} = A$ for all $x \in A$

So $A = \{\Phi\}$ is open set

If $A = X$ as $(X, d)$ is metric space

$x \in X$ $S_r(x) = \{y \in X : d(y, x) < r\}$

clearly $S_r(x) \subseteq X$ for all $x \in X$

So X is a open set.

**Theorem:** Let (X, d) be a metric space. Then every open sphere in X is an open set.

Proof : Let $S_r$ (a) is open spare in X as $S_r(a) \neq \Phi$ so $\exists$ $x \in S_r(a)$

We will produce an open sphere cantered at x and contained in $S_r(a)$

As $x \in S_r(a)$ so $d(x, a) < r$

say $r_1 = r-d(x,a) > 0$

Let $S_{r1}(x)$ be any open sphere with centre x and radius $r_1$

As $S_{r1}(x) \neq \{\Phi\}$ so $\exists$ $y \in S_{r1}(x)$ such that $d(y, x)< r_1$

$$d(x,y) < r-d(x,a)$$
$$d(x,y) + d(x,a) < r$$
$$d(y,a) \leq d(y,x) + d(x,a) < r \text{ as d is metric}$$
$$d(y,a) < r$$
$$d(a,y) < r$$
$$\Longrightarrow \quad y \in S_r(a) \text{ so } S_{r1}(x) \subseteq S_r(a) \text{ for all } x \in S_r(a)$$

Hence $S_r(a)$ is open set.

**Basic results on open set**

**1 :** Let (x, d) be a metric space. A subset G of X is open iff it is the union of open spheres.

**2** : Let (X, d) be a metric space. Then

(i) Arbitrary union of open sets in X is open.

(ii) Finite intersection of open sets in X is open.

(iii) Infinite intersection of open sets may or may not be open. For example, in the usual metric space (R, d) if we take $G_n = (-n, n)$ for all $n \in N$, then each $G_n$ is an open set

**Conclusion:-** Our major focus in this research paper is the study of usual metric space , discrete metric space and open set . The word metric is generalization of ordinary distance, and the concept of metric is not restricted to the sub sets of R, it can be applied to any set. There are many kind of metric spaces some of which play very significant roles in geometry and analysis.

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